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THESIS

EVALUATING THE ANALYTIC HIERARCHY PROCESS AND RECOMMENDED MODIFICATIONS FOR ITS USE IN MULTI-ATTRIBUTE DECISION MAKING

by

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September, 1993

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13. ABSTRACT *(maximum 200 words)*

The Combined Arms Analysis Directorate of the Training and Doctrine Analysis Command (TRAC) uses the Analytic Hierarchy Process (AHP) to evaluate the contribution of modernization initiatives to U.S. Army capabilities. This thesis identifies several problems with using AHP. Most significantly, AHP can cause rank reversal of alternatives if a new alternative is considered, even if the new alternative has the same attribute levels as one of the previous alternatives. This thesis proposes several modifications that would improve results when AHP is used. It contains a different method of weight fitting that appears to provide alternative weights that are more accurate than the traditional AHP eigenvalue method. This thesis has two proposals for improving the nine point integer scale by which pairwise comparisons are made. Most significantly, this thesis proposes a modification to AHP that will maintain a ratio scale and avoid rank reversals. This last improvement requires the decision maker to establish and maintain units of measurement. Additionally, the decision maker must make comparisons of attributes to establish a meaningful scale not sensitive to the abundance or lack of alternatives considered. If units are maintained and the decision maker is consistent in the pairwise comparisons, there will be no rank reversals.

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Evaluating The Analytic Hierarchy Process and Recommended Modifications for its use in Multi-Attribute Decision Making

by

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EXECUTIVE SUMMARY

The Combined Arms Analysis Directorate of the Training and Doctrine Analysis Command (TRAC) uses the Analytic Hierarchy Process (AHP) to evaluate the level of future U.S. Army modernization that is achieved by management decision packages.

A study of AHP shows it contains several flaws. First, and of greatest concern, AHP can allow rank reversals among alternatives if another alternative is considered, or if one is removed. This can occur even if the additional alternative has the same attribute levels a one of the original alternatives.

We use a decision problem which evaluates tanks to illustrate AHP's greatest flaw. Assume that a tank is measured by two attributes, survivability and firepower, and that the decision maker determined firepower is 1.25 times as important as survivability. To maintain that preference and normalize weights to sum to one, weights must be $(\frac{4}{9},\frac{5}{9})$ for survivability and firepower, respectively.

Tank characteristics:

(1) Survivability. On comparing three types of armor, reactive armor is preferred 1.5 over applique, and applique is preferred 2 over rolled homogeneous armor.

(2) Firepower. A tank with a 130mm main gun is preferred 1.4 over a tank with a 120mm main gun. A 120mm main gun is preferred 1.5 over a 105mm main gun. A 105mm main gun is preferred 2.0 over a 90mm main gun.

Initially, we compare two tanks using AHP. Tank 1 has applique armor and a 120mm main gun. Tank 2 has reactive armor and a 105mm main gun.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.4	0.6	0.511
TANK 2	0.6	0.4	0.489

Figure 1. Comparison of two tanks using AHP.

The relative value says one would prefer Tank 1 to Tank 2. That is

<u>Tank 1 > Tank 2</u>.

In addition to the above two tanks, we now consider a third tank. Tank 3 has rolled homogeneous armor and a 130mm main gun. The result of this comparison is disturbing.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.333	0.326	0.329
TANK 2	0.500	0.217	0.343
TANK 3	0.167	0.457	0.328

Figure 2. Comparison of three tanks using AHP.

Even though the decision maker did not change relative weights of firepower or survivability, or the characteristics of the first two tanks, AHP tells the decision maker his preferences have changed. That is

<u>Tank 2 > Tank 1 > Tank 3</u>.

Our modification, illustrated using the same tank problem described above, solves the problem of rank reversal. In our modification, the survivability attribute of Tank 1 is selected as the least preferred alternative in survivability. The firepower attribute of Tank 1 must remain 1.5 times the magnitude of the suvivability attribute. The firepower attribute of Tank 2 (105mm main gun) is determined to be equivalent to the survivability attribute of Tank 1 (applique armor). These restrictions are essential to maintaining a ratio scale.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	1.0	1.5	1.2778
TANK 2	1.5	1.0	1.2222

Figure 3. Modification used on original tank problem.

The result is consistent with the result from using AHP in the original problem. That is

<u>Tank 1 > Tank 2</u>.

Now, we include the third tank, Tank 3.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	1.0	1.5	1.2778
TANK 2	1.5	1.0	1.2222
TANK 3	0.5	2.1	1.3889

Figure 4. Modification with additional alternative included, Tank 3.

We maintain the notional "units" from the two tank problem. That is, Tank 1 survivability is the "unit" for survivability and the equivalency of Tank 1's survivability is maintained with the value of the firepower of Tank 2. The result of this consistency in "units" is the preference

<u>Tank 3 > Tank 1 > Tank 2</u>.

There is no rank reversal.

Also, AHP uses an integer nine point scale, with reciprocals possible, that severely limits the decision maker's ability to evaluate possible alternatives and attributes in a decision problem.

Finally, AHP uses the eigenvalue method of weight fitting with little justification. The weight fitting by the eigenvalue method was consistently (nine cases) less accurate than a simple least squares error (LSE) evaluation method.

Despite the claim of the developer of AHP, Thomas L. Saaty, rank reversals are possible even if the new alternative is not within 10% of original alternatives in attributes. Since rank reversal can occur, AHP cannot maintain a ratio scale, a claim that it makes and that is desirable when output of AHP is used as input to optimization models.

This thesis proposes modifications to AHP that address the above problems. First, it is clear that the restriction on scaling alternatives to integers with a maximum value of nine exacerbates the problems of inconsistency by a decision maker. There are two proposals in this area. First, the decision maker could identify the extreme attributes and extreme alternatives by attribute. Assign the lowest or least preferred a value of one, and the highest or most preferred a value of nine. In this way, all attributes and alternatives will be within the nine point scale. Alternatively, the decision maker could use an open ended scale.

The second proposal for modification of AHP suggests an alternative weight fitting method. The method of least square errors (LSE) for weight fitting produces an exact solution to alternative weights if the decision maker is perfectly consistent, just as AHP's eigenvalue method will. The LSE method appears to have an advantage in the likely case of decision maker inconsistency.

The third modification proposal involves solving the problem of rank reversals while maintaining a ratio scale. The decision maker can assign the most preferred alternative in one attribute a value of one, and assign all other alternatives a value of less than one. The second option is to assign the least preferred alternative in one attribute a value of one. If this standard is maintained and the decision maker is consistent in all pairwise comparisons, there can be no rank reversal.

I. BACKGROUND

A. PROBLEM

The Combined Arms Analysis Directorate of the Training and Doctrine Analysis Command (TRAC) at Fort Leavenworth, Kansas requested assistance in determining values for modernization initiatives. There is concern that the Analytic Hierarchy Process (AHP), the system currently used, might not be the best method to generate relative values of modernization initiatives.

B. SITUATION

The United States Army Training and Doctrine Command (TRAC) at Fort Leavenworth, Kansas developed a new analytical decision support system to help the Army develop a modernization investment program that reflects the values of its senior leadership. TRAC developed the Research Development and Acquisition Alternatives Analyzer (RDA3) to determine values for candidate modernization initiatives by mathematically optimizing allocation of resources.

C. CURRENT SYSTEM

1. Description of RDA3

The RDA3 decision support system involves two distinct phases.

a. Hierarchal Assessment

In the first phase, modernization initiatives (MDEPs) are determined through a hierarchal assessment. The overall goal (Future Army Modernization) is at the top of this hierarchy, and modernization initiatives can be several layers down this hierarchy through layers of supporting goals and objectives, sub-goals and sub-objectives. The goal is the top level of the hierarchy (Future Army Modernization). The first level of the hierarchy consists of 14 Army Mission Areas. Through a series of comparative judgement questions, the respondent or decision maker assesses the importance of each Mission Area to Army modernization. Similarly, sub-areas are assessed for their contribution to Army Mission Areas. This process continues until the bottom level of the hierarchy is reached. The bottom layer consists of Modernization Development Programs (MDEPs), or in some cases, increments of MDEPs. In this way, the decision maker will assess the contribution of each MDEP to the goal of Future Army Modernization.

b. Mathematical Optimization

In the second phase of the RDA3 decision support system, the values of the modernization initiatives (MDEPs), constrained by the annual funding proposal, are mathematically optimized. The mathematical optimization incorporates several decision maker constraints and goals such as annual budgetary limitations, target mission funding levels, and funding of Congressionally or DOD mandated programs. The mathematical optimization currently in use is a new, enhanced math programming algorithm developed by Captain Scott Donahue, USA, in his thesis, while a student at the Naval Postgraduate School under the direction of Professor Rosenthal.

2. Advantages of RDA3

The decision support system, RDA3, possesses many characteristics which contribute to its value as a decision support system. Specifically, RDA3

- is a comprehensive decision support system that considers every MDEP a potential modernization candidate.
- can be executed quickly, with an estimated time of just two to three hours for analysts to examine the outputs received from one iteration and formulate revised inputs that reflect new options to be explored.
- would remain applicable if a new math programming optimization technique was adopted, or if a new method of determining the hierarchal structure was implemented.
- is versatile in that it will provide optimal solutions to budget decisions whether increasing expenditures or reducing spending authorization.
- is highly transportable since it is built primarily on the GAMS linear programming language.
- is used to derive relative values or priorities.
- is controlled by HQDA which provides projected research, development and acquisition (RDA) constraints.

The many advantages of RDA3 make the system of great value to the Army leadership. The system also has the advantage of flexibility in the area in which this thesis recommends change: the hierarchal structure.

3. Modernization Initiatives

Factors that influence the overall desirability of modernization initiatives:

- modernization plans,
- field experience,
- attitudes in the Office of the Secretary of Defense,
- cost,
- Congressional attitudes,
- warfighting contribution,
- history,
- business sense,
- personnel considerations.

These factors, however, are not necessarily an exhaustive list, and there is nothing to prevent the decision maker from reducing the influence of the above factors, or even eliminating one or more factors from consideration entirely.

Given the above factors, TRAC decided the best quantitative scale and measurement process that captures the essence of these influences is the Analytic Hierarchy Process (AHP).

II. THE ANALYTIC HIERARCHY PROCESS

A. SOURCE

The background information in this chapter on the Analytic Hierarchy Process (AHP) is mainly drawn form the creator of AHP, Dr. Thomas L. Saaty. Specifically, the article that provides a description of how to use AHP, and an example of using AHP to buy a house are from "How to make a decision: The Analytic Hierarchy Process" [Ref. 1].

B. HOW TO STRUCTURE A DECISION PROBLEM

Arrange the factors that are important for the decision in a hierarchic structure descending from an overall goal to criteria, subcriteria and alternatives. Include enough detail to:

- thoroughly represent the problem,
- consider environment of problem,
- identify issues or attributes,
- identify participants.

The arrangement of a hierarchy serves two purposes: It gives an overall view of the relationship and helps the decision maker determine if the issues in each level are in the same category so he can make accurate comparisons. Verbal judgements are numerically given values of (1,3,5,7,9) ranging from equal to extreme (equal, moderately more, strongly more, very strongly more, extremely more). These values are the result of comparing the more preferred alternative to the lesser preferred alternative. When comparing a lesser preferred alternative to a more preferred alternative, as expected, the numerical values would be the reciprocals of the above, or $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9})$. The aspect of judgements is discussed in the next section.

C. PAIRWISE COMPARISON SCALING

A primary concern of AHP is with scaling. Saaty states "The Analytic Hierarchy Process is rigorously concerned with the scaling problem and what sort of numbers to use, and how to correctly combine the priorities resulting from them" [Ref. 1:p. 10]. The table Saaty devised to use when making pairwise comparisons is shown in Figure 1 [Ref. 1:p. 15].

Intensity of importance on an absolute scale	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgement strongly (sic) favor one activity over another
5	Essential or strong importance	Experience and judgement strongly favor one activity over another
7	Very strong importance	An activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values between the two adjacent judgements	When compromise is needed
Reciprocals	If activity *i* has one of the above numbers assigned to it when compared with activity *j*, then *j* has the reciprocal value when compared with *i*	
Rationals	Ratios arising from the scale	If consistency were forced by obtaining n numerical values to span the matrix

Figure 1. The AHP Point Scale for Pairwise Comparisons.

D. PAIRWISE COMPARISONS

The aspect of pairwise comparisons that is described below assumes a decision maker is evaluating a problem of m attributes and n alternatives using AHP.

The attributes that contribute to the value of each alternative to be evaluated (Saaty refers to the attributes as "criteria"), are evaluated by pairwise comparison by the decision maker. In an assessment of m attributes, the decision maker is required to perform $\frac{m(m-1)}{2}$ pairwise comparisons. The decision maker would construct a pairwise comparison matrix with m rows and m columns where each row and column represent an attribute. The number of pairwise comparisons is reduced from m^2 by two requirements of AHP. First, the matrix will be reciprocal. That is each element r_{ij} of the pairwise comparison matrix is the reciprocal of r_{ji}, or

$$r_{ij}=\frac{1}{r_{ji}} \qquad \forall i,j=1,\ldots,m.$$

Second, main diagonal elements of the matrix will always have entries of one. That is

$$r_{ii}=1 \quad \forall i=1,\ldots,m.$$

The pairwise comparison procedure is performed at every level of the hierarchy, with the exception of the alternatives themselves, the lowest level of the hierarchy. In other words, if there were one or more levels of subcriteria, pairwise comparisons in the manner described above would be performed.

In the final or lowest level of the hierarchy, the decision maker would make pairwise comparisons of alternatives one attribute at a time. The decision maker would evaluate each alternative by attribute. Ultimately, the decision maker will have m matrices (one for each attribute) of size n×n. Each matrix is formed by making $\frac{n(n-1)}{2}$ pairwise comparisons of the alternatives.

The pairwise comparison matrices are said to be consistent if there is a vector ω of size n, in the case of alternatives (ω would be of size m in the case of attributes), such that

$$r_{ij}=\frac{\omega_i}{\omega_j} \quad \forall i,j=1,\ldots,n. \tag{1}$$

Otherwise, the matrix is not consistent. Note that these equations imply that

$$r_{ij} = r_{ik} \cdot r_{kj}, \quad \forall i, j, k \tag{2}$$

for consistency. The vector ω is made unique by normalizing by dividing by its sum. Thus,

$$\sum_{i=1}^{n} \omega_i = 1 .$$

If we refer to the matrix of pairwise comparisons as **R**; **R** is consistent if, and only if

$$\boldsymbol{R\omega} = n\boldsymbol{\omega} .$$

In a problem where there is some inconsistency present, AHP solves

$$\boldsymbol{R\omega} = \lambda_{max}\boldsymbol{\omega}$$

where λ_{max} is the principal eigenvalue of **R**. This leads to an approximation of ω whose entries correspond to the weights of the alternatives or attributes. To determine the amount of inconsistency and determine if the amount of consistency is acceptable, Saaty developed the consistency index (CI) defined as

$$CI = \frac{(\lambda_{max} - n)}{(n-1)} .$$

Saaty recommends if CI $\leq$ 0.1, accept the estimate of ω. Otherwise, attempt to improve consistency. He does not specifically state what course of action the decision maker should undertake to improve consistency [Ref. 1:p. 13].

The decision maker could be forced into consistency by making just n pairwise comparisons in the first row of the pairwise comparison matrix. The first pairwise comparison would be alternative one compared to itself, which is by definition unity. Excluding this comparison, (n-1) pairwise comparisons are all that are necessary. In this way, the decision maker would obtain the entries for the first row of the pairwise comparison matrix, and define the weights based on those entries

$$(r_{11}, r_{12}, \ldots, r_{1n}) \equiv (\omega_1, \omega_2, \ldots, \omega_n).$$

The entries for the rest of the matrix could be obtained by defining

$$r_{ij} = \frac{\omega_i}{\omega_j} \quad \forall\, i, j = 1, \ldots, n$$

and again normalizing weights to sum to one. With these weights obtained exclusively from the first row the pairwise comparison matrix, every element of the matrix could be obtained. The resulting matrix **R** would be perfectly consistent.

However, AHP, to its credit, does not force this consistency on the decision maker. By requiring $\frac{n(n-1)}{2}$ comparisons, AHP makes the vector ω over-determined and allows inconsistencies.

Therefore, the pairwise comparison matrix could very likely contain inconsistencies. With the presence of inconsistencies, there is no exact solution for the vector ω such that Equation (1) holds for every i and j. The question is how to find an ω that "best" fits these equations when inconsistency is present. With little justification, AHP uses the eigenvalue method previously described. The advantages of this method are (i) if the pairwise comparison matrix is consistent, $\lambda_{max}=n$, and (ii) it allows evaluation of consistency by the consistency index (CI) defined above.

In Chapter IV we discuss and evaluate other methods of determining the vector ω. All methods produce exact results if the decision maker is consistent. We evaluate approximations by defining the error between the pairwise comparison matrix and one produced from the vector of weights, ω.

E. EXAMPLE

In order to illustrate how to use AHP in a problem, attached as Appendix A to this thesis is the example of choosing the best house to buy using AHP from Saaty's article cited above [Ref. 1].

F. INSIGHTS

In light of this chapter's discussion concerning AHP, several questions may have occurred to the reader. For example, given the AHP system of pairwise comparison scaling, how does the decision maker rate three alternatives if the first is judged to be of extreme importance to the second, and the second is determined to be of extreme importance to the third? There is no 81 on the scale, which is what is required of r_{13} if $r_{12}=r_{23}=9$ and Equations (1) and (2) have to hold.

Additionally, how does the restriction to an integer scale compromise the accuracy of pairwise comparisons? Consider three alternatives which result in the following pairwise comparisons. The first is preferred by a factor of three to the second ($r_{12}=3$). The third is preferred by a factor of two to the second ($r_{32}=2$). By how much is the first alternative preferred to the third? Equation (2) shows that for consistency to hold, $r_{13}=1.5$, but that is not on the AHP scale and so could not have been chosen.

These problems and others will be discussed thoroughly in the next chapter.

III. CRITIQUE OF THE ANALYTIC HIERARCHY PROCESS

A. BACKGROUND

The Analytic Hierarchy Process contains several flaws, three of which are discussed in this chapter. Two issues, scaling and ordering, are significant concerns, yet the problem of rank reversal represents a flaw in methodology and is considered the most significant issue regarding the usefulness of AHP. It is for this reason that the problem of rank reversal must be viewed as far more significant than any of the other problems. The result is that AHP can produce rank orderings that are not consistent with the underlying preferences of the decision maker. In Chapter IV, we show how to avoid this *while maintaining the ratio scale*.

1. Rank Reversal

By adding or deleting an alternative, reversal in the ranking of the other alternatives can occur, even though this addition or deletion causes no change in the decision maker's pairwise comparisons. As ironic as it may seem, AHP can tell a decision maker that his or her preference of alternatives would change if an additional alternative were to be considered. Even if the new item considered has the same attribute levels as an original alternative, AHP could tell the decision maker to change original preferences. This is what is described as rank reversal. The concept will be covered in greater depth later in this chapter. This has been much discussed in the literature. See, for example, Schoner and Wedley [Ref. 2], Howard [Ref. 3], Holder [Ref. 4], Roper and Sharp [Ref. 5] and Belton and Gear [Ref. 6]. Saaty maintains that rank reversals are justified because one rates the uniqueness of an alternative. The presence of a duplicate of this alternative must, he claims, decrease the value of the original alternative, even if the decision maker says it does not affect his or her relative strengths of preferences.

2. Pairwise Comparison Scaling

Pairwise comparisons are restricted to a nine point integer scale. This necessarily leads to inconsistency since no attribute or alternative can be 1.5, 4.5 or 10 times more important than another.

3. Weight Fitting

The most appropriate way to determine the weights ω_i from pairwise comparisons may not be the eigenvector method used by AHP. In an article by Hihn and Johnson [Ref. 7], 16 methods are studied, and there appears to be no reason to believe AHP generates the most desirable solutions using the eigenvector technique.

Now we move to a discussion of how AHP is used in practice and how AHP could be changed in a fundamental way. In a problem in which AHP is used, let us assume that there are n attributes indexed i=1,2,...,n with the weight of attribute i given the value ω_i. The attributes will be normalized so that

$$\sum_{i=1}^{n} \omega_i = 1 .$$

We would like to determine a vector of weights $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)^T$ that denotes positive weights obtained by the pairwise comparisons. We will define

$$r_{ij} = \frac{\omega_i}{\omega_j} .$$

This represents the ratio of the weights of attributes i and j which, in Saaty's method, is determined through pairwise comparisons. As was pointed out in Equation (2), if the decision maker is consistent, the ratios will satisfy,

$$r_{ij} = \frac{\omega_i}{\omega_j} = \frac{\omega_i}{\omega_k} \cdot \frac{\omega_k}{\omega_j} = r_{ik} r_{kj} \qquad \forall\, i, j, k.$$

If there is inconsistency, determining the "best" ω is a nontrivial matter. There are n ω_i's to be found using $\frac{n(n-1)}{2}$ r_{ij}'s, so the problem is how to best fit the ω_i's from an over-determined data set.

In AHP, the matrix of these ratios, or comparisons, will approximate the vector ω of the weights by solving the following equation

$$\begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{vmatrix} \begin{vmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{vmatrix} = \lambda_{\max} \begin{vmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{vmatrix} \tag{3}$$

where the matrix of ratios (data from pairwise comparisons) will be labelled **R**. Equation (3) can be expressed in matrix notation as shown below:

$$\boldsymbol{R\omega} = \lambda_{\max}\boldsymbol{\omega}.$$

Characteristics of matrix **R** include

$$r_{ij} = \frac{1}{r_{ji}} \rightarrow \textit{matrix is reciprocal},$$

$$r_{ii} = 1.$$

The diagonal entries of **R** are all one, since a pairwise comparison of an attribute with itself must be one.

Saaty's method forms the matrix **R** by performing $\frac{n(n-1)}{2}$ pairwise comparisons. The next step is to determine the largest eigenvector of **R** and the corresponding right eigenvector. This vector is normalized to give the relative weights. Let $\lambda_{\max}$ represent the largest eigenvalue of **R**. We list the following results found in the AHP literature:

- **R** is reciprocal,
- **R** is not necessarily consistent,

- **R** is consistent if and only if $\lambda_{max}=n$,
- $\lambda_{max}\geq n$.

In AHP, the nine point scale, as previously described, is used to make comparisons. Saaty advises if

$$\frac{\lambda_{max}-n}{n}\leq 0.1,$$

use the results obtained, since comparisons are "consistent enough." However, if

$$\frac{\lambda_{max}-n}{n}>0.1,$$

identify the inconsistencies, modify the comparisons and recalculate. This appears to be a rule-of-thumb with no theoretical foundation.

B. RANK REVERSALS

An axiom of rational decision making implies that addition or deletion of a new alternative, *with no changes in relative preferences for existing alternatives*, should never cause a change in the existing ranking when ranking alternatives. Saaty argues "rank reversals do occur in practice in a way that does not satisfy these assumptions" [Ref. 8:p. 13]. In way of an explanation, Saaty continues, "On seeing too many copies of an attractive alternative, one abandons it for a less attractive one that is unique" [Ref. 8:p. 13]. In an example of rank reversal later in this section, rank reversal of three tanks occurs when a duplicate of one of the tanks is added. Presumably, the decision maker is not planning on fighting a war with one tank and is aware that there are other tanks in existence of all types of tanks considered. In fact, it could be argued that in this case uniqueness of an alternative would be a weakness. In this case, however, AHP rates an alternative as less preferable when a duplicate of that alternative is considered despite the fact that uniqueness is clearly not an advantage.

Saaty shows the frequency of rank preservation of all alternatives for 1000 random cases involving two to nine criteria and two to nine alternatives in distributive, ideal and utility modes of AHP [Ref. 8]. In the distributive mode, weights derived from paired comparisons are normalized to add to 1.0. The ideal mode divides by the weight of the highest ranked alternative for each criterion making the largest weight to be 1.0. The utility mode is a transformation of the ideal mode to an interval scale. The trend in all three cases shows the *rate of rank preservation decreases* as number of criteria and alternatives increase. In other words, *the more complex the problem in terms of criteria and alternatives, the more likely it is that one will have rank reversal.* In the distributive mode with two criteria and two alternatives, there were 963 cases out of 1000 of rank preservation. When nine criteria and nine alternatives were used, rank was preserved just 526 times out of the 1000 cases. Similarly for the ideal mode, 967 cases of rank preservation for two criteria and two alternatives reduced to 832 cases with nine alternatives and nine criteria. Finally for the utility mode, 903 cases of rank preservation with two criteria and two alternatives were reduced to just 664 cases of rank preservation when nine criteria and nine alternatives were used [Ref. 8].

1. Duplication

In 1983, shortly after the introduction of AHP, it was shown that the introduction of a duplicate alternative could, in some cases, cause a rank reversal of the other alternatives [Ref. 6].[1] Saaty responded to this by arguing that rank reversals are justified because an alternative loses some of its appeal if it is not unique. In reality, sometimes a duplicate may reduce the value of an alternative, but in AHP, duplicates of alternatives always reduce the value of the original alternatives. The decision maker *has no choice*. If a duplicate alternative is introduced into the decision problem, the value of the original alternative is reduced. The rank reversal will still occur *even if the uniqueness of an alternative is irrelevant to the decision maker.*

This problem is caused by the particular normalizing of weights in AHP. This is inherent to AHP and cannot be

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[1]The rank reversal example of Belton and Gear is reproduced by Schoner and Wedley in Reference 2.

eliminated without changing the way AHP normalizes weights, which is exactly what we hope the distributive method has done, but the article [Ref. 8] fails to explain the source of the rank reversals. The 474 cases of rank reversal out of the 1000 cases in which nine criteria and nine alternatives were used with the distributive method could be due to inconsistency in pairwise comparisons, or renormalizing weights without regard to units of measurement. We can not be sure if the distributive method is a viable alternative to AHP. There can be two sources of rank reversal. The first source is inconsistency in pairwise comparisons by the decision maker. The second source is from normalizing attributes of alternatives in a way that does not maintain a ratio scale. The former source of rank reversal does not appear to have a solution. The latter source can be solved by either of the modifications described in this thesis. We must caution that Saaty's distributive method might cause many or all of the rank reversals that occurred by a flawed method of normalizing.

2. Near Copies

In response to rank reversals upon the introduction of duplicate alternatives, Saaty, rather than recognize this issue as a potential source of error in the methodology of AHP, suggested elimination of alternatives from consideration that score within 10 percent of another alternative. In an example later in this chapter, the additional alternative introduced, the third tank, is not within 10 percent of either of the first two tanks in either of the two attributes evaluated, *yet a rank reversal still occurs*.

3. Rank Reversal Example

What follows are examples of how the Analytic Hierarchy Process (AHP) can produce a rank reversal in alternatives when an additional alternative is added. It also follows that a rank reversal can occur when an alternative is eliminated from consideration. These examples simulate making a decision on selecting or valuing a weapon system, in this case a tank. However, much like the example of using AHP to compare houses (which is in Appendix A to this thesis), this example is intended to be generic and equally applicable to all ranking or valuing problems. Therefore, the problem of rank reversal is not a special case or exception when AHP is used.

a. *Attributes*

Assume that there are two attributes, survivability and firepower. Further assume that you decide that firepower is 1.25 times as important as survivability.[2] Weights are $(\frac{4}{9}, \frac{5}{9})$ for survivability and firepower, respectively. This

[2]Although the literature on AHP calls for using the scale in Figure 1, the computer application of AHP, "Expert Choice", does allow for a continuous set of comparison numbers to be used.

maintains the weight priorities ($1.25 \times \frac{4}{9} = \frac{5}{9}$), and meets the AHP requirement that weights sum to one. The example of how these attributes satisfy the equality of Equation (3) is shown below where $\lambda_{max} = n = 2$

$$\begin{array}{cc} & \begin{array}{cc} S & F \end{array} \\ \begin{array}{c} S \\ \\ F \end{array} & \begin{vmatrix} 1 & \frac{4}{5} \\ \frac{5}{4} & 1 \end{vmatrix} \end{array} \begin{vmatrix} \frac{4}{9} \\ \frac{5}{9} \end{vmatrix} = 2 \begin{vmatrix} \frac{4}{9} \\ \frac{5}{9} \end{vmatrix}.$$

The labels S and F represent survivability and firepower, respectively. The matrix is consistent, as it must be in any 2×2 reciprocal matrix with entries of one on the main diagonal.

(1) Survivability. On comparing three types of armor, reactive armor is preferred 1.5 over applique, and applique is preferred 2 over rolled homogeneous armor.

(2) Firepower. A tank with a 130mm main gun is preferred 1.4 over a tank with a 120mm main gun. A 120mm main gun is preferred 1.5 over a 105mm main gun. A 105mm main gun is preferred 2.0 over a 90mm main gun.

b. Comparing two Tanks using AHP

Tank 1 has applique armor and a 120mm main gun. Tank 2 has reactive armor and a 105mm main gun. So Tank 1 has more firepower but is less survivable than Tank 2.

The consistent matrix equation satisfying Equation (3) from the pairwise comparisons of Tank 1 with Tank 2 for the attribute of survivability is shown below:

$$\begin{array}{c|cc|} & T1 & T2 \\ T1 & 1 & \frac{2}{3} \\ T2 & \frac{3}{2} & 1 \end{array} \begin{vmatrix} \frac{4}{10} \\ \frac{6}{10} \end{vmatrix} = 2 \begin{vmatrix} \frac{4}{10} \\ \frac{6}{10} \end{vmatrix}.$$

Similarly, the matrix equation satisfying Equation (3) from the pairwise comparisons of Tank 1 with Tank 2 for the attribute of firepower is shown below:

$$\begin{array}{c|cc|} & T1 & T2 \\ T1 & 1 & \frac{3}{2} \\ T2 & \frac{2}{3} & 1 \end{array} \begin{vmatrix} \frac{6}{10} \\ \frac{4}{10} \end{vmatrix} = 2 \begin{vmatrix} \frac{6}{10} \\ \frac{4}{10} \end{vmatrix}.$$

Figure 2 shows the method used to determine relative value of alternatives once the weights of attributes have been assigned and pairwise comparisons of alternatives have been made.

	ATTRIBUTES		RELATIVE VALUE
TANK	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.4	0.6	0.511
TANK 2	0.6	0.4	0.489

Figure 2. Comparison of two tanks using AHP.

The figure shows Tank 1 is preferred to Tank 2, or

$$\boxed{\text{Tank 1} > \text{Tank 2.}}$$

If we let $\boldsymbol{R_{2T}}$ be the pairwise comparison matrix obtained by pairwise comparisons of the relative value of the tanks, we obtain the following matrix:

$$\boldsymbol{R_{2T}} = \begin{array}{c} \\ T1 \\ T2 \end{array} \begin{array}{c} \begin{array}{cc} T1 & T2 \end{array} \\ \begin{vmatrix} 1 & 1.045 \\ 0.957 & 1 \end{vmatrix} \end{array}.$$

c. *Additional Alternative*

Tank 1 has applique armor and a 120mm main gun, and Tank 2 has reactive armor and a 105mm main gun. These are the same tanks as in the section above. Tank 3 has rolled homogeneous armor and a 130mm main gun. Tank 2 has the most survivability and Tank 3 has the most firepower. Tank 3 has the least survivability and Tank 2 has the least firepower. Tank 1 was rated between Tanks 2 and 3 in both attributes.

	ATTRIBUTES		RELATIVE VALUE
TANK	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.333	0.326	0.329
TANK 2	0.500	0.217	0.343
TANK 3	0.167	0.457	0.328

Figure 3. Comparison of three tanks using AHP.

When Tank 3 is included, *Tank 2 becomes preferred to Tank 1* which is in turn preferred to Tank 3. Figure 3 shows Tank 2 is preferred to Tank 1, which is preferred to Tank 3, or

$$\boxed{\text{Tank 2} > \text{Tank 1} > \text{Tank 3.}}$$

Note that this reversal happened even though no changes were made to the relative importance of the options in either survivability or firepower, or between these two attributes. If we let $\boldsymbol{R}_{3T}$ be the pairwise comparison matrix obtained by pairwise comparisons of the relative values of the three tanks, we obtain the following matrix:

$$\boldsymbol{R}_{3T} = \begin{array}{c} \\ T1 \\ T2 \\ T3 \end{array} \begin{array}{c} \begin{array}{ccc} T1 & T2 & T3 \end{array} \\ \left| \begin{array}{ccc} 1 & 0.959 & 1.003 \\ 1.043 & 1 & 1.046 \\ 0.997 & 0.956 & 1 \end{array} \right| \end{array}.$$

If the ratio scale had been maintained, the pairwise comparisons between Tank 1 and Tank 2 should be the same as they were in $\boldsymbol{R}_{2T}$. This is not the case.

d. *Explanation of Rank Reversal*

In the original formulation that compared two tanks, the dilution of components of relative value can be seen in a comparison of the two tables showing component relative values before and after Tank 3 was included. Originally, the component in the table contributing the largest amount to either tank's relative value is represented by the firepower attribute of Tank 1. This can be read off the table as 0.333. This is significantly higher than the relative value component of survivability of Tank 2 which is only 0.2666. With the addition of Tank 3 which has a large rating for firepower, the relative value components of firepower for the two original tanks (Tank 1 and Tank 2) are significantly reduced. Once Tank 3 is added as an alternative, all components of relative attributes of the original two tanks are reduced. In this case, Tank 3 was rated higher in firepower than survivability. Therefore, the firepower rating of Tank 1 (originally 0.333 then 0.181 for a reduction of 0.152) had a greater reduction in magnitude than the survivability rating of Tank 2 (originally 0.2666 then 0.222 for a reduction of only 0.0446). The greater reduction in magnitude of the firepower attribute of Tank 1 led to the rank reversal with Tank 2.

e. Duplication of Alternative

Now let us suppose a fourth tank is to be considered. The fourth tank has the same characteristics as Tank 2. Therefore, since it also has reactive armor, it will be preferred by 3 to 1 over Tank 3 in survivability, which has rolled homogeneous armor. It is preferred by 1.5 to 1 over Tank 1 with survivability characteristic of applique type armor. Similarly in firepower, its small diameter main gun is least preferred. Both attributes must be renormalized and multiplied by the weight of the attribute to obtain the new relative value of the tanks. This is done in Figure 4.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.2222	0.2679	0.2475
TANK 2	0.3333	0.1783	0.2472
TANK 3	0.1112	0.3755	0.2581
TANK 4	0.3333	0.1783	0.2472

Figure 4. Comparison of four tanks using AHP.

The decision maker might consider the uniqueness of a tank to be irrelevant, but nevertheless, AHP produces a dramatic rank reversal in this case. Recall that originally AHP produced a preference order of two tanks as Tank 1 > Tank 2. With three tanks, the order becomes Tank 2 > Tank 1 > Tank 3. Now by adding a fourth tank with the same characteristics as Tank 2, the preference of the four tanks is 3, 1, 2, 4, that is Tank 3 > Tank 1 > Tank 2 = Tank 4 (**the "worst" becomes the "best"!**).

f. Discussion of Results

The reversal in preference order of the first two tanks by the addition of a third is caused by flawed logic in AHP and the disregard for measurement units. Even though there are no natural units for measuring survivability and firepower, one must define an underlying scale based on a standard. Both the scale intervals and standard can be arbitrarily chosen, but once they are, measurements must be made consistent with this scale.

C. PAIRWISE COMPARISON SCALING

There is inherent inconsistency in the nine point scale used by AHP. The nine point pairwise comparison scale is defined in Figure 1 in Chapter II.

Immediately, it is possible to pose theoretical problems impossible to formulate when restricted to this nine point scale. This is the case if one is faced with three alternatives, A, B, and C with the following pairwise comparison results. Alternative C is deemed to have extreme importance over B, and B is deemed to have extreme importance over A. One would assume C would have much more than extreme importance over A, but this is not possible with AHP. To say that alternative C is just of extreme importance over A defies logic. That would imply that the decision maker should be indifferent between B and A (after all, C is of extreme importance over both these alternatives). Perhaps the decision maker should now decide alternative C now only has strong importance to B and B only has strong importance to A so C can have extreme importance over A. This new hierarchy is neither consistent, nor does it explain why a decision maker should revise his or her estimate of the pairwise comparison of alternatives.

The situation above was entirely theoretical with generic alternatives. But what if the problem was travel from the west coast to the east coast, and alternative A was walking, alternative B was driving a car and alternative C was flying a plane. It seems obvious alternative C would be preferred by more than a factor of nine to alternative A.

Inconsistency is not the exception with this nine point scale. Almost any time there are more than two or three comparisons to be made, the scale will almost always lead to inconsistencies. To maintain consistency, the pairwise comparisons of alternatives should be the product of the scale number determined. Take, for example, four alternatives, A, B, C, and D, that are deemed of moderate importance over their preceding alternative. Therefore, alternative A, being the least preferred alternative, would receive a 1 in a pairwise comparison to itself. Alternative B would receive a 3 in comparison to A (moderately preferred). Similarly, alternative C receives a score of 3 compared to B, or 9. But what score does D receive? It is moderately preferred to C, but C already has the highest score allowed on the nine point scale.

In theory, there should be no limit to the scale. The scale should be open ended, and not restricted to a maximum value of nine.

One simple way to avoid the scaling problems described above would by to insure the decision maker stayed within the scale. The decision maker would identify the extreme attributes, that is the most and least preferred attributes. The least preferred attribute would be defined as one on the scale, and the most preferred would be defined as nine. All other attributes would, therefore, fall somewhere in between these two extremes and all attributes would be within the scale.

D. WEIGHT FITTING

In any evaluation of attributes that allows inconsistency, the user must somehow be able to determine how much inconsistency there is in the evaluation. Given a matrix **R** = $[r_{ij}]$, suppose the weights ω have been obtained. One expression that measures the error between the weights and the ratios is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(r_{ij} - \frac{\omega_i}{\omega_j}\right)^2 .$$

A second and more tractable error measure is

$$Error = \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega_j r_{ij} - \omega_i)^2 . \tag{4}$$

Both of these expressions are zero if, and only if, **R** is consistent. One definition of the "best" weights ω is that they minimize the expression in Equation (4) for a given matrix **R**. This is the method of least squared error (LSE) for evaluation comparisons, and was the first method we selected to evaluate preference fitting. Without further comment, we point out that the above equation is not the unique way to evaluate consistency. In fact, the article by Hihn and Johnson [Ref. 7] lists and describes 16 methods of finding the "best" $\boldsymbol{\omega}$ for a given $\mathbf{R}$ for various definitions of "best". It appears that none of the methods differ from each other a great deal, but Saaty's method was one of the poorest.

Saaty' method, the second method we evaluated by using preference fitting, is to solve

$$\boldsymbol{R} = \lambda_{max}\boldsymbol{\omega}$$

where λ_{max} represents the largest eigenvalue of the matrix $\mathbf{R}$. In all cases, $\lambda_{max} \geq n$, and when $\mathbf{R}$ is consistent, $\lambda_{max} = n$. Hihn and Johnson point out that Saaty's justification for selecting this particular method, the right eigenvector technique, is that if $\mathbf{R}$ is consistent, this technique produces the solution $\lambda_{max} = n$. But they also point out that all of the 16 methods discussed in Reference 7 produce this exact solution in the case of consistency. Therefore, the only method of differentiating the 16 techniques is to evaluate each technique's results when inconsistency is present. Hihn and Johnson show that Saaty's technique is one of the poorest of the 16 techniques evaluated.

The third and final method we studied in the area of preference fitting we call the dimension method. In this method, we solve the following equation

$$\boldsymbol{R\omega} = n\boldsymbol{\omega}.$$

1. **Approach**

In each of the three methods, the error matrix was evaluated based on the LSE formula. This gave a value for the total error of the method used. The value of this total error does not necessarily prove any one method is better than another. We did, however, obtain interesting results using Saaty's pairwise comparison matrices from his example of using AHP to buy a house [Ref. 1].

2. **Results**

A 3×3 matrix of attributes was evaluated using Saaty's right eigenvector method, the LSE method and what we call the dimension method. The matrix is reproduced below with results of pairwise comparisons as entries. As required, each main diagonal entry is a one, and the matrix is reciprocal.

$$\boldsymbol{R} = \begin{vmatrix} 1 & 6 & 8 \\ \frac{1}{6} & 1 & 4 \\ \frac{1}{8} & \frac{1}{4} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 6 & 8 \\ 0.17 & 1 & 4 \\ 0.13 & 0.25 & 1 \end{vmatrix}. \tag{5}$$

Using Saaty's method, the priority vector of weights was determined as $(0.754, 0.181, 0.065)^T$ with an error of 17.82 (λ_{max}=3.136). We will denote the matrix obtained by AHP as

$\boldsymbol{R}_{SAATY}$. The result using least squares was a priority vector of weights of $(0.727, 0.202, 0.071)^T$ and error of 7.47. We will denote the matrix obtained by the LSE method as $\boldsymbol{R}_{LSE}$. Finally, the dimension method produced a priority vector of weights of $(0.768, 0.145, 0.087)^T$ with an error of 12.31. We will denote the matrix obtained by the dimension method as $\boldsymbol{R}_{DIM}$. We point this out not as an evaluation of any technique, but simply to demonstrate that different techniques produce different results. Naturally, the LSE method had the smallest error. Next was the dimension method, and finally, with the highest error of any of the three methods, was Saaty's eigenvalue method. The reader should compare the following three matrices with the one in Equation (5) and decide which of these is "closest" to **R**. Note that, as expected, the LSE method fits the larger numbers in **R** better than the smaller ones.

$$\boldsymbol{R}_{SAATY} = \begin{vmatrix} 1 & 4.16 & 11.6 \\ 0.24 & 1 & 2.78 \\ 0.09 & 0.36 & 1 \end{vmatrix} \qquad \boldsymbol{R}_{LSE} = \begin{vmatrix} 1 & 3.60 & 10.24 \\ 0.28 & 1 & 2.84 \\ 0.10 & 0.35 & 1 \end{vmatrix}$$

$$\boldsymbol{R}_{DIM} = \begin{vmatrix} 1 & 5.29 & 8.83 \\ 0.28 & 1 & 2.84 \\ 0.11 & 0.6 & 1 \end{vmatrix}$$

These results were from using these methods in one of the pairwise comparison matrices for the three alternatives in just one of the eight attributes considered. We were curious to see how each method compared if used in the overall problem. The results of an APL program used to evaluate errors by method for alternative pairwise comparison matrices are in Appendix C to this thesis. Each of the eight 3×3 matrices represents one of the attributes considered. Each matrix was used to determine a vector ω of weights for the alternatives and each method was evaluated for the error term. Naturally, since the criterion used is minimum least squares, the LSE method had the smallest error. In all cases except one, the dimension method had the next smallest error. In seven of the eight cases, Saaty's eigenvalue method had the largest error.

The program in Appendix C also includes the 8×8 pairwise comparison matrix to approximate the vector ω of weights for each attribute, what Saaty refers to as level one of the hierarchy. The error term for this matrix showed the LSE method produced an error of 237.75. Saaty's eigenvalue method gave the next smallest error at 259.06, and the dimension method had by far the highest error term of 858.40.

IV. SUGGESTED MODIFICATIONS TO AHP

A. DESCRIPTION OF MODIFICATION

The two significant problems with AHP identified in the previous chapter, rank reversals and scaling, are addressed in this chapter. The modification we recommend for AHP is one that will avoid rank reversal. It can also avoid the problem of AHP's nine point scale.

1. Ratio Modification

The decision maker, through pairwise comparisons, forms a matrix of ratios of pairwise comparisons. If this matrix changes when additional alternatives are considered, of course rank reversals can be expected. If the decision maker elects to reassess his or her previous alternatives' values, this is a separate matter entirely. Without the decision maker changing his or her original preferences, the ranking (and matrix of ratios) should remain unchanged.

2. Scale Modification

The most extreme comparison of alternatives should be used to establish the scaling for other alternatives. There is nothing wrong with the original nine point scale, or a scale from 1-10, but all numbers between one and the scale extreme would have to be included. Lacking this modification, the scale should be open ended to allow comparisons of alternatives differing by orders of magnitude in value to the decision maker.

Recall that AHP requires two types of comparisons and normalization of weights assigned. The first comparison involves comparing the importance of one attribute with another. This phase of implementing AHP is unchanged by this modification. The other type of comparison is to determine the weight of each alternative's characteristics by attribute. This phase of AHP should be modified so that the least preferred alternative in an attribute is assigned a value of one (or the most preferred). By doing this, the renormalized weights obtained by pairwise comparison of attributes *will maintain a ratio scale and avoid rank reversals.*

3. Weight Fitting

There seems to be no credible reason not to use the least square error (LSE) method to fit weights to alternatives or attributes. The examples in Hihn and Johnson [Ref. 7] are convincing. The eigenvalue method used by AHP is mediocre, at best, among the 16 techniques evaluated for weight fitting. The justification for use of the eigenvalue method in AHP is that an exact solution is obtained in the case of consistency. But exact solutions are produced by all 16 techniques in the case of consistency.

B. REASON FOR MODIFICATION

The Analytic Hierarchy Process (AHP) is currently used to obtain coefficients of the optimization program for the math programming algorithm used by the Director, Program Analysis and Evaluation to maximize future Army modernization subject to a funding constraint.

To assess the desirability of a management decision package (MDEP), TRAC decided to use the AHP as a quantitative scale and measurement process. This thesis shows that AHP will provide coefficients for the optimization algorithm that are not measurable and can be subject to rank reversal if certain alternatives are added or deleted from the original formulation. Even if rank reversal does not occur, how can one have any confidence in the ranking measures when they change due to no changes in the preferences of the user. The idea of using such a flawed system for making major investment decisions is very disturbing. This thesis determines a way to include some aspects of AHP, but not suffer from rank reversal problems.

C. APPROACH

Each MDEP can be converted into its relative contribution to Future Army Moderni: . ion through multiple levels of hierarchy. Although the MDEP's contribution to Future Army Modernization may be a meaningless value to a decision maker, it will be consistent in its relationship to other MDEPs because all MDEPs will be expressed in the same "units" of Future Army Modernization. Additionally, if it is necessary to add new alternatives, the original alternatives will maintain the same values of coefficients for the math programming algorithm.

This procedure is consistent for multiple level problems. The alternatives will receive the same value whether the problem is solved with all levels used, or if the problem is reduced to a single level by cross multiplying weights for each level.

This thesis shows that any model of alternatives and attributes can be expanded to include other attributes without encountering any problem with rank reversal.

1. Additional Attributes

The weight of each attribute will remain the same. Elements of each attribute will not be renormalized to sum to one if additional alternatives are introduced, or if original alternatives are eliminated. In fact, even in the original formulation, weights of alternatives by attribute will not be normalized to sum to one. This implies that the value of each original alternative will not change because the value of an alternative is the weight of each attribute times the value of the alternative's attribute summed over all attributes. The relative value of alternatives should not change if more or less alternatives are considered.

In the original formulation, all alternatives by attribute will be scaled by one of two methods. First, the decision maker can assign a value of one to the most preferred alternative for one attribute. We call this the "standard attribute." In other attributes, alternatives are assigned a value based on the decision maker's value comparisons between attributes. The decision maker must create a link between attributes by declaring an equivalency between an alternative of one attribute and an alternative in other attributes. One can not arbitrarily decide that the decision maker places equal value on all most preferred alternatives in every attribute and assign each a value of one. The most preferred alternative in other attributes could be "better" (assigned a value greater than one) or "worse" (assigned a value of less than one) when compared to the decision maker's "standard attribute" value of one. New alternatives are simply assigned values based on the original formulation's value in each attribute. If the new attribute is preferred by a factor of three to the "standard attribute" from the original formulation, it will be assigned a value of three. If the new alternative's attribute is less preferred than the original formulation's most preferred attribute, the (0,1) scale will be maintained. Even in the original formulation, however, the elements of other attributes may not fit a (0,1) scale.

When using the second method, the decision maker can assign a value of one to the lowest or least preferred alternative for one attribute, which we will refer to as the "standard attribute." In this attribute, new alternatives are simply assigned values based on the original formulation's least preferred attribute values of one. In other attributes, alternatives are assigned a value based on the decision maker's value comparisons between attributes.

2. Applicability

The modification to AHP discussed in this proposal would be a significant improvement to the method used by PAE to obtain coefficients to the math programming algorithm to optimize future Army modernization. This method would insure no rank reversals in alternatives whether alternatives are duplicated, eliminated, or completely new alternatives are added. In addition, the values of each alternative will have consistent units that can allow comparisons of magnitude.

D. THE TANK PROBLEM REVISITED

1. Description of Approaches

This section demonstrates the use of our suggested modification to AHP in two ways. Both avoid rank reversals that plagued our earlier examples.

a. Link between Attributes

To establish and maintain a ratio scale and avoid rank reversals, the decision maker must compare the elements of different attributes and determine a relative value between a specific element of one attribute and an element of other attributes. It is essential to have a link between attributes. Otherwise, an alternative with an extremely undesirable attribute would give overwhelming value to other alternatives simply by assigning the least preferred alternative a value of one. We assume the decision maker, given the weights of $(\frac{4}{9}, \frac{5}{9})$ for survivability and firepower, respectively, assigns the same value to survivability of applique armor and firepower of a 105mm main gun. We illustrate this concept by a simple example.

We assume the decision maker has a tank with attributes of applique armor and a 105mm main gun, and could improve attributes. His or her priority for doing so would be equal to the weights of the attributes. That is, the decision maker would have preference for improving firepower of 1.25 times his or her preference for improving survivability. This illustrates a serious flaw with AHP.

The weights of attributes cannot be independent of alternatives. In our case, would the decision maker have the same priorities given a tank with rolled homogeneous armor and a 130mm main gun? The decision maker possessing a tank with the lowest survivability attribute (rolled homogeneous armor) and the highest firepower attribute (130mm main gun) should be more concerned about improving survivability.

In this modification, the survivability rating could be tripled by replacing the rolled homogeneous armor with reactive armor. This will contribute $\frac{4}{9}$ of the survivability improvement to the tank's relative value. Although the firepower attribute will contribute $\frac{5}{9}$ of its improvement to the relative value of the tank, the firepower rating is already the highest obtainable. It would almost certainly be nearly impossible to triple the firepower rating. Therefore, improving the survivability attribute would contribute a greater increase to the relative value of the tank than increasing firepower. Despite the lower weight for the survivability attribute, a decision maker will elect to improve survivability over firepower if he or she possessed a tank with a high firepower rating and low survivability rating.

With this modification, weights are not independent of alternatives. Unfortunately, AHP cannot make this claim. We can now consider our units of measurement to be in terms of applique armor and a 105mm main gun. We will give this "unit" a label of SAPF105.

b. First Approach

First, of the original alternatives considered, the lowest, or least desirable alternative is given a rating of one in one of the attributes considered. That alternative becomes the standard to determine attribute values for all other attributes, and we call that the "standard attribute."

Attribute weights are normalized in a way that this modification does not change (that is, the sum of the weights of the attributes will still be equal to one). In other attributes, ratings could be much lower than one. These ratings, or scores, will depend on the "standard attribute" and will be scaled accordingly. An additional alternative could be even less preferred in an attribute than the originally least preferred alternative, so it is possible to have a value of less than one in the attribute containing the "standard attribute" once additional alternatives are considered.

c. *Second Approach*

The second approach rates the highest, or most preferable alternative in one of the attributes as one. Once additional attributes are considered, relative values of higher than one are possible in the "standard attribute." This could occur if the new alternative had a more preferable rating in the "standard attribute" than the original formulation's most preferable alternative in the "standard attribute."

2. Tank Problem Solved by First Approach

This section solves the tank problem introduced earlier to demonstrate rank reversal in a way that avoids rank reversals.

a. *Original Formulation*

The original two tanks are evaluated in Figure 5 using the approach that gives the tank with the least desirable characteristic in one attribute a score of one. In this case, survivability of Tank 1 is the "standard attribute" from which other values are derived. The decision maker has determined equivalency between survivability characteristic of applique armor and firepower of a 105mm main gun. The result is a different relative value for each tank, but still Tank 1 > Tank 2.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	1.0	1.5	1.2778
TANK 2	1.5	1.0	1.2222

Figure 5. First approach with original formulation.

The relative value scores are not meaningless. The relative value of a tank is in terms of survivability of Tank 1 and firepower of Tank 2, because Tank 2 happens to have a firepower rating of 1.0. Its firepower attribute, a 105mm main gun is equivalent to the "standard attribute." A theoretical "worst" tank with Tank 1's survivability (applique armor) and Tank 2's firepower (105mm gun) would receive a relative value of 1.0. Tank 1's relative value of 1.28 means it is 1.28 times more desirable than this theoretical tank. Recall that we developed the label SAPF105 for the "units" of measurement of tanks in terms of survivability attribute applique armor and firepower attribute of 105mm main gun. Tank 1 is "worth" 1.28 SAPF105s, and Tank 2 is "worth" 1.22 SAPF105s.

b. Additional Alternative Considered

The original two tanks, and the third tank, are evaluated in Figure 6. Again, the least desirable characteristic in one attribute (survivability) receives a score of one. Once an additional alternative is considered, it is possible for an alternative to receive a relative value of less than one in survivability. This could happen if the new alternative had attributes less desirable than the original problem's least preferable alternative in survivability (Tank 1). The result of considering the additional alternative is the same relative value scores for the first two tanks.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	1.0	1.5	1.2778
TANK 2	1.5	1.0	1.2222
TANK 3	0.5	2.1	1.3889

Figure 6. First approach with third tank.

Tank 3 has the highest relative value. Therefore, the decision maker's preference would be Tank 3 > Tank 1 >

Tank 2. There was no rank reversal between the first two tanks.

Unlike AHP, this modification (i) did not have rank reversal, (ii) maintained a ratio scale of relative value.

c. *Duplication of Alternative*

A fourth tank is now added. Just as in the previous example, it has the same characteristics as Tank 2. These four tanks are evaluated in Figure 7.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	1.0	1.5	1.2778
TANK 2	1.5	1.0	1.2222
TANK 3	0.5	2.11	1.3889
TANK 4	1.5	1.0	1.2222

Figure 7. First approach with fourth tank.

The first three tanks receive the same relative value scores as they did when considered previously. The preferences for these four tanks are Tank 3 > Tank 1 > Tank 2 = Tank 4. There is no rank reversal among the original alternatives.

Adding the fourth tank (i) did not cause rank reversal, (ii) maintained a ratio scale of relative value.

d. Analysis

If we let $\boldsymbol{R}'_{2T}$ be the pairwise comparison matrix obtained by pairwise comparisons of the relative values of the two tanks, we obtain the following matrix:

$$\boldsymbol{R}'_{2T} = \begin{array}{c} \\ T1 \\ T2 \end{array} \begin{array}{c} \begin{array}{cc} T1 & T2 \end{array} \\ \begin{vmatrix} 1 & 1.05 \\ 0.96 & 1 \end{vmatrix} \end{array}.$$

If we let $\boldsymbol{R}'_{4T}$ be the pairwise comparison matrix obtained by the pairwise comparisons of the relative values of the four tanks, we obtain the following matrix:

$$\boldsymbol{R}'_{4T} = \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array} \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \begin{vmatrix} 1 & 1.05 & 0.92 & 1.05 \\ 0.96 & 1 & 0.88 & 1 \\ 1.09 & 1.14 & 1 & 1.14 \\ 0.96 & 1 & 0.88 & 1 \end{vmatrix} \end{array}.$$

The upper left 2×2 entries of the matrix are identical to the matrix $\boldsymbol{R}'_{2T}$. Therefore, ***the ratio scale was maintained***. Rank reversal will not occur because the relative value of alternatives will not change.

3. Tank Problem Solved by Second Approach

This section solves the tank problem using the second approach described earlier. The highest or most preferred alternative in one attribute receives a score of one. In this case, we select Tank 2's survivability attribute as the "standard attribute" by which other alternatives within that attribute will be measured. Let us again assume that the decision maker, in order to "link" the two attributes, has determined indifference between Tank 1's survivability (applique armor) and the firepower of Tank 2 (105mm main gun). Since this is the same equivalency established in the first method, we should expect to maintain the same ratio scale for this method, if it is to have merit.

a. Original Formulation

The original two tanks are evaluated in Figure 8 using the second approach. The result is a different relative value score for each tank, but still Tank 1 > Tank 2. Essentially, what we have done is change "units." Now tanks are evaluated on the basis of Tank 2's survivability (reactive armor) and Tank 1's firepower (120mm main gun). We will label these "units" SRAF120.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.67	1.0	0.8519
TANK 2	1.0	0.67	0.8148

Figure 8. Second approach with original formulation.

The relative values obtained by this method have meaning. A theoretical "best" tank with the highest possible rating in each attribute (this would be a tank with reactive armor and a 120mm gun) would receive a score of one. Tank 1's relative value of 0.85 means it is 0.85 times as desirable as this theoretical tank. From units derived above, Tank 1 is worth 0.85 SRAF120s.

b. Additional Alternative Considered

The original two tanks, and the third tank, are evaluated in Figure 9. Again, the most desirable characteristic in one attribute (survivability) receives a score of one. It is now possible for the new alternative (Tank 3) to receive a relative value score of greater than one in survivability. A rating of one is based on survivability of reactive armor. The result of considering the additional alternative is the same relative value scores for the first two tanks.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.67	1.0	0.8518
TANK 2	1.0	0.67	0.8148
TANK 3	0.33	1.4	0.9259

Figure 9. Second approach with third tank.

This method also avoided the rank reversal encountered in the original AHP formulation of this problem.

> This modification
> (i) avoided the rank reversal of AHP,
> (ii) maintained a ratio scale of relative value.

c. Duplication of Alternative

A fourth tank, with the same characteristics as Tank 2 is added. These four tanks are evaluated in Figure 10.

TANK	ATTRIBUTES		RELATIVE VALUE
	SURVIVABILITY (4/9)	FIREPOWER (5/9)	
TANK 1	0.67	1.0	0.8518
TANK 2	1.0	0.67	0.8148
TANK 3	0.33	1.4	0.9259
TANK 4	1.0	0.67	0.8148

Figure 10. Second approach with fourth tank.

Unlike the example of using AHP with an additional alternative added having the same characteristics as an original alternative, this method avoids rank reversal. The reason for this is the relative value of an alternative will not change unless the decision maker makes a conscious decision to change weights of alternatives or the rating of an alternative because of reassessment or new information.

> Adding the fourth tank
> (i) did not cause rank reversal,
> (ii) maintained a ratio scale of relative value.

d. Analysis

If we let $\boldsymbol{R}^{-}_{2T}$ be the pairwise comparison matrix obtained by pairwise comparisons of the relative values of the two tanks in the original problem, we obtain the following matrix:

$$\boldsymbol{R}^{-}_{2T} = \begin{array}{c} \\ T1 \\ T2 \end{array} \begin{array}{cc} T1 & T2 \\ \left|\begin{array}{cc} 1 & 1.05 \\ 0.96 & 1 \end{array}\right| \end{array}.$$

If we let $\boldsymbol{R}^{-}_{4T}$ be the pairwise comparison matrix obtained by pairwise comparisons of the relative values of the four tanks using the second approach, we obtain the following matrix:

$$\boldsymbol{R}^{-}_{4T} = \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \end{array} \begin{array}{c} \begin{array}{cccc} T1 & T2 & T3 & T4 \end{array} \\ \left|\begin{array}{cccc} 1 & 1.05 & 0.92 & 1.05 \\ 0.96 & 1 & 0.88 & 1 \\ 1.09 & 1.14 & 1 & 1.14 \\ 0.96 & 1 & 0.88 & 1 \end{array}\right| \end{array}.$$

The upper left 2x2 matrix is identical to the 2x2 matrix obtained in the original formulation from $\boldsymbol{R}^{-}_{2T}$. This shows that this approach maintains a ratio scale when additional alternatives are considered.

4. Conclusion

Both approaches maintain a ratio scale of alternatives and avoid rank reversals. Because $\boldsymbol{R}'_{4T}=\boldsymbol{R}^{-}_{4T}$, either approach solves the rank reversal problems identified previously when using AHP. Whether the decision maker decides to rank the highest alternatives in one attribute as one, or the lowest as one, he or she will obtain identical answers. That is, the hierarchy of preferences will be identical and the ratios of comparisons of alternatives will be identical.

V. CONCLUSION

There are at least three modifications which should be made to AHP before it is used in a decision making problem.

A. PAIRWISE COMPARISON SCALING

The nine point scale used in AHP is overly restrictive. There are at least two ways to modify scaling procedures that eliminate a significant potential for inconsistency.

1. Decision Maker Defined Scale

Instead of using the value of nine to signify extreme importance of one attribute over another (or one element of an alternative over another), the decision maker should define the nine point scale based on the characteristics of the problem. We suggest defining the most preferred alternative as nine, and the least preferred alternative as one. All intermediate alternatives must fall within the scale. Similarly, when making pairwise comparisons of alternatives by attribute, the extreme alternatives should be used to define the scale.

2. Open Ended Scale

The decision maker could alternatively use an open ended scale for pairwise comparisons. If the decision maker feels there are several orders of magnitude of difference between alternatives, he or she could use a multiplicative open ended scale to determine preferences.

B. WEIGHT FITTING

There appears to be no advantage to using AHP's eigenvalue method of weight fitting, other than its ease of evaluation by the AHP consistency index. But since the consistency index standard of 0.9 is itself arbitrary, evaluating one's decision making by the consistency index is, in reality, not much of an advantage. The least square error (LSE) method of weight fitting, not surprisingly, produces a smaller error. Of course, LSE was used to determine the error of both methods.

C. RANK REVERSAL

We have demonstrated two ways to avoid rank reversals. Both methods depend on the decision maker to establish and maintain a system of "units" in which relative values of alternatives will be measured.

1. First Method

The decision maker selects the least preferred alternative in one of the attributes considered. This least preferred alternative, which we call the "standard attribute" is assigned a value of one. All other alternatives within that attribute are assigned values based on each alternative's comparison to the least preferred alternative. The decision maker must also relate the value of the "standard attribute" to other attributes. This is an essential step to maintaining a ratio scale. The decision maker will decide a ratio between the "standard attribute" and an element within each of the other attributes. No matter how the problem is changed from this point, that ratio must remain the same. The alternatives must not be renormalized with the addition or deletion of alternatives.

2. Second Method

The decision maker selects the most preferred alternative in one of the attributes considered. This most preferred alternative, which is the "standard attribute" is assigned a value of one. All other alternatives within that attribute are assigned values based on each alternative's comparison to the most preferred alternative. The decision maker conducts identical steps as performed in the first method to relate values to other attributes.

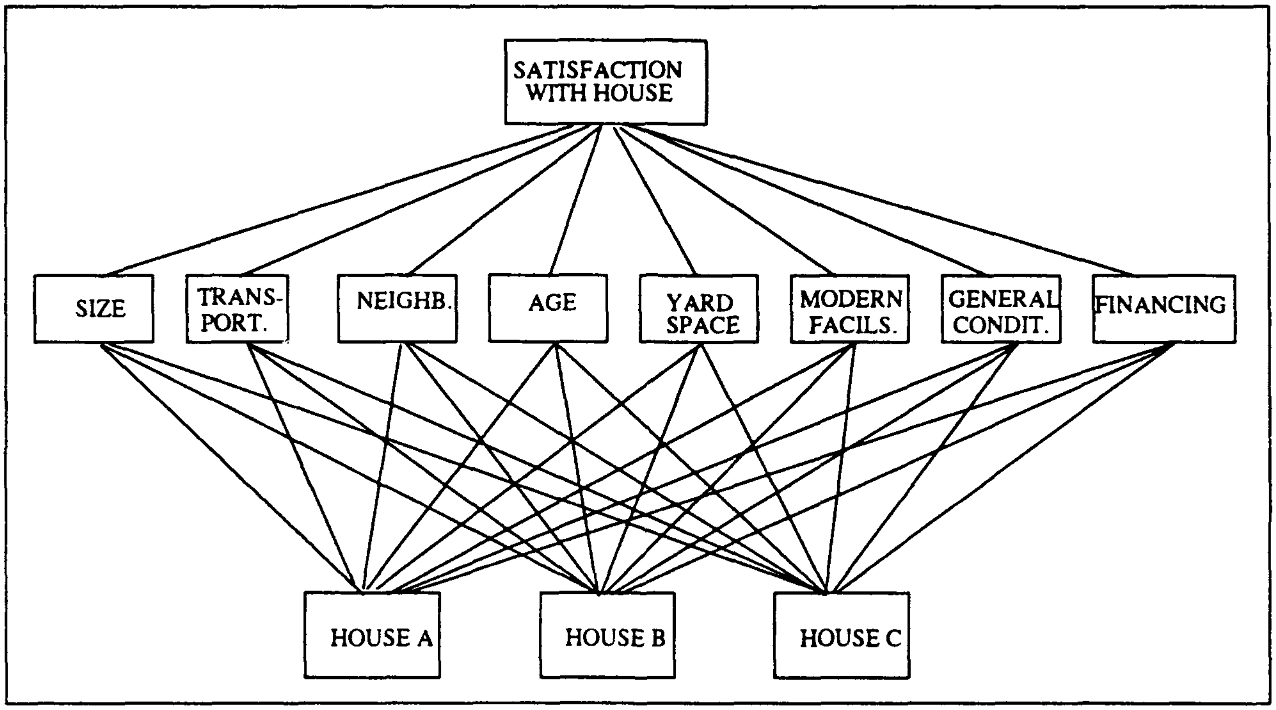
3. Results

Both methods produced identical pairwise comparison matrices at every step of the tank example problem. Neither method will generate rank reversals when the decision maker is consistent. We demonstrated use of invented units as labels for the magnitude of the relative value in each example of the use of each method. Since the notional units were consistently used to measure the relative value of alternatives as additional alternatives were considered, the relative value of alternatives never changed. Both methods avoid rank reversals and maintain a ratio scale.

APPENDIX A

AHP AND CHOOSING A HOUSE

This example is taken from T. L. Saaty, "How to make a decision: The Analytic Hierarchy Process", European Journal of Operations Research, Vol. 48, pages 9 - 26, 1990. The purpose is to compare three houses, A, B, and C, using eight attributes to find the most preferred house.



The following matrix shows a pairwise comparison of all eight (8) attributes, together with the vector of weights produced by AHP that add to 1.0.

	1	2	3	4	5	6	7	8	PRIORITY VECTOR
1	1	5	3	7	6	6	1/3	1/4	.173
2	1/5	1	1/3	5	3	3	1/5	1/7	.054
3	1/3	3	1	6	3	4	6	1/5	.188
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	.018
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6	.031
6	1/6	1/3	1/4	4	2	1	1/5	1/6	.036
7	3	5	1/6	7	5	5	1	1/2	.167
8	4	7	5	8	6	6	2	1	.333

A Matrix and Priority Vector for House Ranking Problem

The following eight (8) matrices show the results of pairwise comparisons of the three houses on each attribute.

SIZE

	A	B	C	PRIORITY VECTOR
A	1	6	8	.754
B	1/6	1	4	.181
C	1/8	1/4	1	.065

EV=3.136

YARD SPACE

	A	B	C	PRIORITY VECTOR
A	1	5	4	.674
B	1/5	1	1/3	.101
C	1/4	3	1	.226

EV=3.068

TRANSPORTATION

	A	B	C	PRIORITY VECTOR
A	1	7	1/5	.233
B	1/7	1	1/8	.054
C	5	8	1	.713

EV=3.247

MODERN FACILITIES

	A	B	C	PRIORITY VECTOR
A	1	8	6	.747
B	1/8	1	1/5	.060
C	1/6	5	1	.193

EV=3.197

NEIGHBORHOOD

	A	B	C	PRIORITY VECTOR
A	1	8	6	.745
B	1/8	1	1/4	.065
C	1/6	4	1	.181

EV=3.136

GENERAL CONDITION

	A	B	C	PRIORITY VECTOR
A	1	1/2	1/2	.200
B	2	1	1	.400
C	2	1	1	.400

EV=3.000

AGE OF HOUSE

	A	B	C	PRIORITY VECTOR
A	1	1	1	.333
B	1	1	1	.333
C	1	1	1	.333

EV=3.000

FINANCING

	A	B	C	PRIORITY VECTOR
A	1	1/7	1/5	.072
B	7	1	3	.650
C	5	1/3	1	.278

EV=3.065

LOCAL AND GLOBAL PRIORITIES

	1	2	3	4	5	6	7	8	
	(.173)	(.054)	(.188)	(.018)	(.031)	(.036)	(.167)	(.333)	
A	.754	.233	.754	.333	.674	.747	.200	.072	.396
B	.181	.055	.065	.333	.101	.060	.400	.650	.341
C	.065	.713	.181	.333	.226	.193	.400	.278	.263

From these AHP shows that House A is preferred to B is preferred to C.

APPENDIX B

What follows is a demonstration of how to determine the least squares approximation of determining weights of attributes. This demonstration was performed on a 3×3 matrix, but the reader should be able to recognize the pattern that develops.

Recall that if the pairwise comparisons are perfectly consistent, the following relationship holds.

$$r_{ij} = \frac{\omega_i}{\omega_j}, \qquad r_{ij} \text{ given in pairwise comparisons.}$$

The error for the comparisons

$$e(\omega) = \sum_{i=1}^{n} \sum_{j=1}^{n} (\omega_i - \omega_j r_{ij})^2 .$$

By definition, main diagonal entries must be one, or $r_{ii}=1 \ \forall i$. Expanding the error term for a 3×3 matrix

$$e = (\omega_1 - \omega_2 r_{12})^2 + (\omega_1 - \omega_3 r_{13})^2 + (\omega_2 - \omega_1 r_{21})^2 +$$
$$(\omega_2 - \omega_3 r_{23})^2 + (\omega_3 - \omega_1 r_{31})^2 + (\omega_3 - \omega_2 r_{32})^2 .$$

The reason the above expression contains only six terms is because the entries for i=j are all zero.

The row entries of the solution matrix are the respective partial derivatives of the error with respect to ω_i, where i designates the row number, and setting the equation equal to zero. For the first row,

$$\frac{\partial e}{\partial \omega_1} = 2(\omega_1 - \omega_2 r_{12}) + 2(\omega_1 - \omega_3 r_{13}) - 2r_{21}(\omega_2 - \omega_1 r_{21}) - 2r_{31}(\omega_3 - \omega_1 r_{31})$$

$$2\omega_1 = -\omega_1 r_{21}^2 - \omega_1 r_{31}^2 + \omega_2(r_{12} - r_{21}) + \omega_3(r_{31} - r_{13})$$

$$3\omega_1 = \omega_1 - \omega_1 r_{21}^2 - \omega_1 r_{31}^2 + \omega_2(r_{12} - r_{21}) + \omega_3(r_{31} - r_{13}) \ .$$

The last step, adding ω_1 to each side, was done in light of the solution we are seeking, $\mathbf{E}\omega = n\omega$, where $\mathbf{E}$ is the matrix of errors, ω is the vector of weights and n is the dimension of this vector. Similarly, the partial derivatives with respect to ω_2 and ω_3 can be determined, which lead to the following matrix representing the solution form of matrix $\mathbf{E}$.

$$\begin{vmatrix} (1 - r_{21}^2 - r_{31}^2) & (r_{12} + r_{21}) & (r_{13} + r_{31}) \\ (r_{12} + r_{21}) & (1 - r_{12}^2 - r_{32}^2) & (r_{23} + r_{32}) \\ (r_{13} + r_{31}) & (r_{23} + r_{32}) & (1 - r_{13}^2 - r_{23}^2) \end{vmatrix}$$

To determine the error when using the dimension method, we solve the equation

$$\boldsymbol{R\omega} = n\boldsymbol{\omega} \ .$$

Recall that in the case of inconsistency, there is no exact solution to this equation. There are many ways to approximate the vector ω. The procedure for deriving the dimension method's approximation, obtained by multiplying both sides by an identity matrix of size n

$$\boldsymbol{R}\boldsymbol{\omega} = n\boldsymbol{I}\boldsymbol{\omega} \rightarrow (n\boldsymbol{I}-\boldsymbol{R})\,\boldsymbol{\omega} = 0 \,.$$

Expanding,

$$\begin{vmatrix} n & & 0 \\ & \ddots & \\ 0 & & n \end{vmatrix} \cdot \begin{vmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{vmatrix} - \begin{vmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & \ddots & & \\ \vdots & & \ddots & \\ r_{n1} & & & r_{nn} \end{vmatrix} \cdot \begin{vmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{vmatrix} = 0 \,.$$

This is equivalent to:

$$\begin{matrix} n\omega_1 - \sum_{j=1}^{n} r_{1j}\,\omega_j \\ n\omega_2 - \sum_{j=1}^{n} r_{2j}\,\omega_j \\ \vdots \qquad \vdots \\ n\omega_n - \sum_{j=1}^{n} r_{nj}\,\omega_j \end{matrix} = 0 \,.$$

This leads to a final expression for the error matrix for the dimension method of estimating weights shown below

$$\boldsymbol{R} - \begin{vmatrix} \dfrac{\omega_1}{\omega_1} & \cdots & \dfrac{\omega_1}{\omega_n} \\ \vdots & & \vdots \\ \dfrac{\omega_n}{\omega_1} & \cdots & \dfrac{\omega_n}{\omega_n} \end{vmatrix} = \begin{vmatrix} e_{11} & \cdots & e_{1n} \\ \vdots & & \vdots \\ e_{n1} & \cdots & e_{nn} \end{vmatrix} .$$

APPENDIX C

SIZEM COMPARE WSSIZE

METHOD	ERROR	WEIGHTS		
SAATY :	17.820	.754	.181	.065
DIMENSION :	12.314	.727	.202	.071
LEAST SQRS :	7.477	.780	.136	.084

TRANSM COMPARE WSTRANS

METHOD	ERROR	WEIGHTS		
SAATY :	35.902	.233	.055	.712
DIMENSION :	33.357	.199	.060	.740
LEAST SQRS :	29.833	.193	.064	.743

NEIGHM COMPARE WSNEIGH

METHOD	ERROR	WEIGHTS		
SAATY :	17.820	.754	.065	.181
DIMENSION :	12.314	.727	.071	.202
LEAST SQRS :	7.477	.780	.084	.136

AGEM COMPARE WSAGE

METHOD	ERROR	WEIGHTS		
SAATY :	.000	.333	.333	.333
DIMENSION :	.000	.333	.333	.333
LEAST SQRS :	.000	.333	.333	.333

YARDM COMPARE WSYARD

METHOD	ERROR	WEIGHTS		
SAATY :	4.439	.674	.101	.226
DIMENSION :	3.627	.655	.105	.240
LEAST SQRS :	3.026	.697	.118	.185

MODFACM COMPARE WSMODFAC

METHOD ERROR WEIGHTS

SAATY : 27.540 .747 .060 .193

DIMENSION : 17.584 .707 .067 .226

LEAST SQRS : 14.946 .784 .078 .138

GENCONM COMPARE WSGENCON

METHOD ERROR WEIGHTS

SAATY : .000 .200 .400 .400

DIMENSION : .000 .200 .400 .400

LEAST SQRS : .000 .200 .400 .400

FINANCEM COMPARE WSFINANCE

METHOD ERROR WEIGHTS

SAATY : 5.860 .072 .650 .278

DIMENSION : 8.334 .068 .654 .278

LEAST SQRS : 5.438 .087 .670 .243

HOUSEATTM

1.000	5.000	3.000	7.000	6.000	6.000	0.333	0.250
0.200	1.000	0.333	5.000	3.000	3.000	0.200	0.143
0.333	3.000	1.000	6.000	3.000	4.000	6.000	0.200
0.143	0.200	0.167	1.000	0.333	0.250	0.143	0.125
0.167	0.333	0.333	3.000	1.000	0.500	0.200	0.167
0.167	0.333	0.250	4.000	2.000	1.000	0.200	0.167
3.000	5.000	0.167	7.000	5.000	5.000	1.000	0.500
4.000	7.000	5.000	8.000	6.000	6.000	2.000	1.000

HOUSEATTM COMPARE WSHOUSEATT

METHOD	ERROR	WEIGHTS						
SAATY:259	.173	.054	.188	.018	.031	.036	.167	.333
DIMENSION:858	-.031	.081	.258	.022	.043	.052	.170	.404
LSE:237	.279	.447	.098	.027	.041	.041	.072	.395

LIST OF REFERENCES

1. Saaty, Thomas L., *How to make a decision: The Analytic Hierarchy Process*, European Journal of Operational Research, Vol. 48, (1990) 9-26.

2. Schoner, Bertram and Wedley, William C., *Ambiguous Criteria Weights in AHP: Consequences and Solutions*, Decision Sciences, Vol. 20, (1989) 462-475.

3. Howard, Ronald A., *Heathens, Heretics and Cults: The Religious Spectrum of Decision Aiding*, Interfaces, Vol. 22, No. 6, (1992) 15-27.

4. Holder, R. D., *Some Comments on the Analytic Hierarchy Process*, Journal of Operational Research Society, Vol. 41, No. 11, (1990) 1073-1076.

5. Roper-Lowe, G. C. and Sharp, J. A., *The Analytic Hierarchy Process and its Application to Information Technology Decision*, Journal of Operational Research Society, Vol. 41, (1990) 49-59.

6. Belton, V. and Gear, T., *On a shortcoming of Saaty's Method of Analytic Hierarchies*, Omega, Vol. 11, (1983) 228-230.

7. Hihn, Jairus M., and Johnson, Charles R., *Evaluation Techniques for Paired Ratio-Comparison Matrices in a Hierarchical Decision Model*, Measurement in Economics, Physica-Verlag Heidelberg, (1988) 269-288.

8. Saaty, Thomas L. and Vargas, L. G., *Experiments on Rank Preservation and Reversal in Relative Measurement*, Mathematical Computational Modeling, Vol. 17, (1993) 13-18.

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